

FINDING MSR OF A GIVEN GRAPH OF AT MOST SEVEN VERTICES BY GIVING VECTOR REPRESENTATIONS

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ABSTRACT. In this paper, we study the minimum rank among positive semidefinite matrices with a given graph of at most seven vertices (msr) by giving vector representations.

Key words. rank, positive semidefinite, graph of a matrix

AMS subject classification: 15A18, 15A57, 05C50

1. INTRODUCTION

Given a connected graph G with n vertices, we associate to G a set $\mathcal{H}(G)$ of Hermitian $n \times n$ matrices by the following way,

$$\mathcal{H}(G) = \{A \mid A = A^*, a_{ij} \neq 0 \text{ for } i \neq j \text{ if and only if } (i, j) \text{ is an edge of } G\}$$

where A^* is the complex conjugate of A and a_{ij} is the ij -entry of A . We define $\mathcal{P}(G)$ be the subset of $\mathcal{H}(G)$ whose members are positive semidefinite matrices. For a given graph G , $\mathcal{P}(G)$ is non-empty because laplacian matrix $L(G) \in \mathcal{P}(G)$. Define

$$msr(G) = \min_{A \in \mathcal{P}(G)} \text{rank}(A)$$

To introduce the result of this paper, we need the following preparation.

Definition 1.1. An induced subgraph H of a graph G is obtained by deleting all vertices except for the vertices in a subset S . For a graph G , we consider its tree size, denoted $ts(G)$, which is the number of vertices in a maximum induced tree.

Lemma 1.1. [2, Lemma 2.1] *If H is an induced subgraph of a connected graph G , then $msr(H) \leq msr(G)$.*

Given a simple connected graph G , we can define the tree size $ts(G)$ of G . It is known that for a tree T , $msr(T) = ts(T) - 1$. Hence by Lemma 1.1, we get $ts(G) - 1 \leq msr(G)$.

Lemma 1.2. [2, Corollary 3.5] *If a simple connected graph G has a pendant vertex v , which is simply vertex of degree 1, then $msr(G) = msr(G - v) + 1$.*

2. FINDING MSR BY GIVING VECTOR REPRESENTATIONS

Suppose G is a connected graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$. We call a set of vectors $V = \{\vec{v}_1, \dots, \vec{v}_n\}$ in \mathbb{C}_m a vector representation (or orthogonal representation) of G if for $i \neq j \in V(G)$, $\langle \vec{v}_i, \vec{v}_j \rangle \neq 0$ whenever i and j are adjacent in G and $\langle \vec{v}_i, \vec{v}_j \rangle = 0$ whenever i and j are not adjacent in G . For any matrix B , $A = B^*B$ is a semi-definite

positive matrix and $\text{rank}(A) = \text{rank}(B)$. For any semi-definite positive matrix A , there exists a matrix B such that $A = B^*B$ and $\text{rank}(A) = \text{rank}(B)$.

For a large family of graphs, [2] has given a good way to get msr. But [2] did not give the msr problem completely. In this section we solve the msr for graphs with at most seven vertices but does not satisfies the conditions of graphs in [2]. For the readers convenience, we list all these graphs here: $G706, G710, G817, G836, G864, G867, G870, G872, G876, G877, G946, G954, G955, G979, G982, G992, G997 - G1000, G1003 - G1007, G1053, G1056, G1060, G1065, G1069, G1084, G1089 - G1097, G1100, G1101, 1104, G1105, G1123, G1125, G1135, G1145, G1146, G1148, G1149, G1152 - G1157, G1159, G1160, G1165, G1167, G1168, G1170, G1176, G1179, G1189, G1191, G1194 - G1197, G1199, G1200, G1202, G1205, G1207 - G1212, G1222, G1224, G1228, G1230, G1231, G1233, G1241, G1242, G1248, G1250$. We give the msr of all those graphs by finding the vector representations.

In the following examples, for vectors v_i and v_j , by abusing notations, we write $\langle v_i, v_j \rangle = v_i \cdot v_j$. If $v_i \cdot v_j = 0$, then we write $v_i \perp v_j$.

Example 2.1.

$$\text{msr}(G706) = 5$$

The tree size of $G706$ is 5. First let's show $\text{msr}(G706) > 4$. Let $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ be a vector representation of $G706$. It follows that $v_1 \perp \text{span}\{v_3, v_4, v_5, v_6\}$, $v_2 \perp \text{span}\{v_5, v_6\}$, $v_3 \perp \text{span}\{v_4, v_5, v_6\}$, and $v_7 \perp \text{span}\{v_4, v_5\}$. Hence $v_5 \notin \text{span}\{v_6\}$. Otherwise $v_6 \cdot v_7 = 0$. Similarly, $v_4 \notin \text{span}\{v_5, v_6\}$. Otherwise $v_2 \cdot v_4 = 0$. Similarly, $v_3 \notin \text{span}\{v_4, v_5, v_6\}$. Otherwise $v_3 = 0$. If the $\text{msr}(G706)$ is 4, we get $v_1 \in \text{span}\{v_3, v_4, v_5, v_6\}$. It follows that $v_1 = 0$. This is a contradiction.

On the other hand, let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 4 & 1 & 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 2 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & 1 \\ 1 & 2 & 1 & 0 & 0 & 1 & 3 \end{bmatrix}$$

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Then $B^T B = A$, $A \in \mathcal{P}(G706)$ and $\text{rank}(B) = 5$.

Example 2.2.

$$\text{msr}(710) = 5$$

The tree size of $G710$ is 5. First let's show $\text{msr}(G710) > 4$.

Suppose $\text{msr}(G710) = 4$. Let $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ be a vector representation. Then $v_1 \perp \text{span}\{v_3, v_4, v_5, v_6\}$, $v_2 \perp \text{span}\{v_4, v_5, v_6\}$, $v_3 \perp \text{span}\{v_5, v_7\}$, $v_7 \perp \text{span}\{v_4, v_5\}$. It

follows that v_3, v_4, v_5, v_6 must be linearly dependent. If v_4, v_5, v_6 are linearly independent, then $v_3 \in \text{span}\{v_4, v_5, v_6\}$. Hence $v_2 \cdot v_3 = 0$. This is a contradiction. Hence v_4, v_5, v_6 are linearly dependent. We know v_4 and v_5 are linearly independent because $v_3 \cdot v_4 \neq 0$ and $v_3 \cdot v_5 = 0$. It follows that $v_6 \in \text{span}\{v_4, v_5\}$. It follows that $v_6 \cdot v_7 = 0$. This is a contradiction.

On the other hand, let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 2 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 3 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & -1 & 1 & 3 & 6 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Then $B^T B = A$, $A \in \mathcal{P}(G710)$ and $\text{rank}(B) = 5$.

Example 2.3.

$$msr(G817) = 4$$

The tree size of $G817$ is 5 and let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 5 & 2 & 0 & 0 & 0 & 3 \\ 0 & 2 & 2 & 2 & 0 & 1 & 0 \\ 0 & 0 & 2 & 5 & 1 & 2 & -1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 & 0 & 1 & -1 \\ 0 & 3 & 0 & -1 & 1 & -1 & 4 \end{bmatrix}$$

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 & 0 & 1 & -1 \end{bmatrix}$$

Then $B^T B = A$, $A \in \mathcal{P}(G817)$ and $\text{rank}(B) = 4$.

Example 2.4.

$$msr(G836) = 4$$

The tree size of $G836$ is 5 and let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 3 & 1 & 1 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 2 & -1 \\ 0 & 0 & 3 & 1 & 2 & 5 & -2 \\ 1 & 2 & 0 & 0 & -1 & -2 & 3 \end{bmatrix}$$

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 2 & -1 \end{bmatrix}$$

Then $B^T B = A$, $A \in \mathcal{P}(G836)$ and $\text{rank}(B) = 4$.

Example 2.5.

$$msr(G864) = 4$$

The tree size of $G864$ is 5 and let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 & -2 \\ 1 & 5 & 2 & 0 & 0 & -1 & 0 \\ 0 & 2 & 3 & 1 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 & 2 & 3 \\ 1 & -1 & 0 & 0 & 2 & 4 & 0 \\ -2 & 0 & 3 & 1 & 3 & 0 & 10 \end{bmatrix}$$

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 1 & 1 & 1 & 2 \\ 0 & 2 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 & -1 & -1 \end{bmatrix}$$

Then $B^T B = A$, $A \in \mathcal{P}(G864)$ and $\text{rank}(B) = 4$.

Example 2.6.

$$msr(G867) = 4$$

The tree size of $G867$ is 5 and let

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 7 & 5 & 0 & -1 & 0 & 3 \\ 0 & 5 & 5 & 0 & 0 & 0 & 3 \\ 1 & 0 & 0 & 7 & 1 & 2 & -1 \\ 0 & -1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 2 & 1 & 2 & 0 \\ 0 & 3 & 3 & -1 & 0 & 0 & 2 \end{bmatrix}$$

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & -2 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 2 & 2 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Then $B^T B = A$, $A \in \mathcal{P}(G867)$ and $\text{rank}(B) = 4$.

Example 2.7.

$$msr(G870) = 4$$

The tree size of $G870$ is 5 and let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 2 & -2 \\ 1 & 4 & 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 2 & 1 & 1 & 5 \\ 0 & 0 & 0 & 1 & 2 & -2 & 0 \\ 2 & -2 & 0 & 1 & -2 & 14 & 0 \\ -2 & 0 & 4 & 5 & 0 & 0 & 22 \end{bmatrix}$$

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 2 & -2 \\ 0 & 1 & 1 & 1 & 0 & 0 & 4 \\ 0 & -1 & 0 & 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & 0 & -1 & 3 & 1 \end{bmatrix}$$

Then $B^T B = A$, $A \in \mathcal{P}(G870)$ and $\text{rank}(B) = 4$.

Example 2.8.

$$msr(G872) = 4$$

The tree size of $G872$ is 5 and let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 4 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 4 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 & 4 & 2 \\ 0 & 0 & 0 & 1 & 0 & 2 & 2 \end{bmatrix}$$

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & -1 & 1 & 1 \end{bmatrix}$$

Then $B^T B = A$, $A \in \mathcal{P}(G872)$ and $\text{rank}(B) = 4$.

Example 2.9.

$$msr(G876) = 4$$

The tree size of $G876$ is 5 and let

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 & 3 & 2 & 0 \\ 1 & 0 & 3 & 0 & 0 & -1 & 2 \\ 1 & 0 & 0 & 6 & 0 & -1 & -3 \\ 0 & 3 & 0 & 0 & 9 & 3 & 1 \\ 0 & 2 & -1 & -1 & 3 & 2 & 0 \\ 0 & 0 & 2 & -3 & 1 & 0 & 3 \end{bmatrix}$$

Let

$$B = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 2 & 1 & -1 \\ 0 & 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & -2 & 1 & 1 & 1 \end{bmatrix}$$

Then $B^T B = A$, $A \in \mathcal{P}(G876)$ and $\text{rank}(B) = 4$.

Example 2.10.

$$msr(G877) = 4$$

The tree size of $G877$ is 5.

Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & -1 & 1 \\ 1 & 3 & 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 & -5 \\ 0 & 0 & 1 & 1 & 0 & -1 & -3 \\ 0 & 3 & 1 & 0 & 5 & 1 & 0 \\ -1 & 0 & 0 & -1 & 1 & 3 & 0 \\ 1 & 0 & -5 & -3 & 0 & 0 & 14 \end{bmatrix}$$

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 1 & 0 & -1 & -3 \\ 0 & 1 & 0 & 0 & 2 & 0 & 1 \end{bmatrix}$$

Then $B^T B = A$, $A \in \mathcal{P}(G877)$ and $\text{rank}(B) = 4$.

Example 2.11.

$$msr(G946) = 4$$

The tree size of $G946$ is 3. First let's show that $msr(G946) > 3$.

Let $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ be a vector representation of $G946$. We have $v_1 \perp \{v_3, v_4, v_5, v_7\}$, $v_2 \perp \{v_4, v_5\}$, $v_3 \perp \{v_6\}$, $v_4 \perp \{v_6, v_7\}$. If $msr(G946) = 3$, then $v_3 = av_4 + bv_7$ for some nonzero a and b . Hence $0 = av_4 \cdot v_6 + bv_7 \cdot v_6 = bv_7 \cdot v_6$. This is a contradiction.

On the other hand, let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & -1 & 0 \\ 1 & 2 & 1 & 0 & 0 & -2 & 1 \\ 0 & 1 & 3 & 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 2 & 1 & 2 \\ -1 & -2 & 0 & 0 & 1 & 3 & 1 \\ 0 & 1 & 3 & 0 & 2 & 1 & 5 \end{bmatrix}$$

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 2 \end{bmatrix}$$

Then $B^T B = A$, $A \in \mathcal{P}(G946)$ and $\text{rank}(B) = 4$.

Example 2.12.

$$msr(G954) = 4$$

The tree size of $G954$ is 4.

Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 15 & 5 & 1 & 0 & -1 & 6 \\ 0 & 5 & 2 & 0 & 0 & -1 & 2 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 3 & 2 & 0 \\ 1 & -1 & -1 & 1 & 2 & 3 & 0 \\ 1 & 6 & 2 & 0 & 0 & 0 & 3 \end{bmatrix}$$

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 3 & 1 & 0 & -1 & -1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Then $B^T B = A$, $A \in \mathcal{P}(G954)$ and $\text{rank}(B) = 4$.

Example 2.13.

$$msr(G955) = 4$$

The tree size of $G955$ is 5.

Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 6 & 3 & 0 & -1 & 3 & -2 \\ 0 & 3 & 3 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & -1 & 0 & 0 & 2 & -1 & 1 \\ 1 & 3 & 2 & 1 & -1 & 3 & 0 \\ 0 & -2 & 0 & 1 & 1 & 0 & 2 \end{bmatrix}$$

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & -1 & 1 & -1 \end{bmatrix}$$

Then $B^T B = A$, $A \in \mathcal{P}(G955)$ and $\text{rank}(B) = 4$.

Example 2.14.

$$msr(G979) = 4$$

The tree size of $G979$ is 5.

Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 3 & 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 3 & -1 & 2 & 0 \\ 0 & 0 & 0 & -1 & 1 & -1 & 1 \\ 0 & 1 & 1 & 2 & -1 & 2 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 & 3 \end{bmatrix}$$

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 & 1 & -1 & 1 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 \end{bmatrix}$$

Then $B^T B = A$, $A \in \mathcal{P}(G979)$ and $\text{rank}(B) = 4$.

Example 2.15.

$$msr(G982) = 4$$

The tree size of $G982$ is 5.

Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 4 & 2 & 0 & 0 & 6 & 2 \\ 0 & 2 & 2 & 4 & 0 & 4 & 0 \\ 0 & 0 & 4 & 26 & -2 & 2 & -6 \\ 0 & 0 & 0 & -2 & 2 & -2 & 2 \\ 0 & 6 & 4 & 2 & -2 & 14 & 0 \\ 1 & 2 & 0 & -6 & 2 & 0 & 4 \end{bmatrix}$$

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 3 & -1 & 3 & -1 \\ 0 & 1 & 0 & -4 & 0 & 2 & 1 \end{bmatrix}$$

Then $B^T B = A$, $A \in \mathcal{P}(G982)$ and $\text{rank}(B) = 4$.

Example 2.16.

$$msr(G992) = 4$$

The tree size of $G992$ is 5.

Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 2 \\ 1 & 2 & 1 & 0 & 0 & 3 & 1 \\ 0 & 1 & 2 & 1 & 0 & 4 & 0 \\ 0 & 0 & 1 & 2 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 & 1 & 2 & -1 \\ 1 & 3 & 4 & 4 & 2 & 13 & 0 \\ 2 & 1 & 0 & 0 & -1 & 0 & 7 \end{bmatrix}$$

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 & 2 & -1 \\ 0 & 1 & 1 & 0 & 0 & 2 & -1 \end{bmatrix}$$

Then $B^T B = A$, $A \in \mathcal{P}(G992)$ and $\text{rank}(B) = 4$.

Example 2.17.

$$msr(G997) = 4$$

The tree size of $G997$ is 5.

Let

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 & 1 & 1 \\ 2 & 5 & 1 & 0 & 0 & 3 & 1 \\ 0 & 1 & 2 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & -1 & 1 \\ 1 & 3 & 2 & 0 & -1 & 4 & 0 \\ 1 & 1 & 0 & 2 & 1 & 0 & 4 \end{bmatrix}$$

Let

$$B = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & -1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & -1 \end{bmatrix}$$

Then $B^T B = A$, $A \in \mathcal{P}(G997)$ and $\text{rank}(B) = 4$.

Example 2.18.

$$msr(G998) = 4$$

The tree size of $G998$ is 3. First let's show that $msr(G998) > 3$.

Let $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ be a vector representation of $G998$. We have $v_1 \perp \{v_3, v_4, v_6\}$, $v_2 \perp \{v_4, v_6\}$, $v_3 \perp \{v_5\}$, $v_4 \perp \{v_5\}$, $v_5 \perp \{v_7\}$, $v_6 \perp \{v_7\}$. If $msr(G998) = 3$, then $av_3 + bv_4 + cv_6 = 0$ for some nonzero a , b and c . Hence $0 = av_3 \cdot v_5 + bv_4 \cdot v_5 + cv_6 \cdot v_5 = cv_6 \cdot v_5$. We get $c = 0$. Similarly, from $0 = av_3 \cdot v_2 + bv_4 \cdot v_2$, we get $a = 0$. It turns out that $b = 0$. This is a contradiction.

On the other hand, let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & -1 & 0 & -4 \\ 1 & 2 & 1 & 0 & -2 & 0 & -3 \\ 0 & 1 & 2 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 2 & 0 & 3 & -1 \\ -1 & -2 & 0 & 0 & 4 & -1 & 0 \\ 0 & 0 & 1 & 3 & -1 & 5 & 0 \\ -4 & -3 & -1 & -1 & 0 & 0 & 22 \end{bmatrix}$$

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & -1 & 0 & -4 \\ 0 & 0 & 1 & 1 & 1 & 1 & -2 \\ 0 & 0 & 0 & 1 & -1 & 2 & 1 \\ 0 & 1 & 1 & 0 & -1 & 0 & 1 \end{bmatrix}$$

Then $B^T B = A$, $A \in \mathcal{P}(G998)$ and $\text{rank}(B) = 4$.

Example 2.19.

$$msr(G999) = 4$$

The tree size of $G999$ is 5.

Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 2 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 5 & 2 & 0 & 2 & -1 \\ 0 & 0 & 2 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & -1 & 1 \\ 1 & 1 & 2 & 0 & -1 & 3 & 0 \\ 1 & 0 & -1 & 1 & 1 & 0 & 3 \end{bmatrix}$$

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & -1 \end{bmatrix}$$

Then $B^T B = A$, $A \in \mathcal{P}(G999)$ and $\text{rank}(B) = 4$.

Example 2.20.

$$msr(G1000) = 3$$

The tree size of $G1000$ is 4.

Let

$$B = \begin{bmatrix} 1 & -5 & 0 & 0 & 0 & 1 & 2 \\ 0 & -2 & 1 & 1 & 0 & -2 & 1 \\ 0 & 1 & 0 & 2 & 1 & 1 & 0 \end{bmatrix}$$

Let

$$A = B^T B$$

Then $A \in \mathcal{P}(G1000)$ and $\text{rank}(A) = 3$.

Example 2.21.

$$msr(G1003) = 4$$

The tree size of $G1003$ is 5.

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 1 & 0 & 1 & -2 \\ 0 & 0 & 3 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & -1 & 1 \end{bmatrix}$$

Let

$$A = B^T B$$

Then $A \in \mathcal{P}(G1003)$ and $\text{rank}(A) = 4$.

Example 2.22.

$$msr(G1004) = 4$$

The tree size of $G1004$ is 5.

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 1 & 0 & 1 & -1 \\ 0 & 0 & 2 & 0 & 1 & -1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & -1 \end{bmatrix}$$

Let

$$A = B^T B$$

Then $A \in \mathcal{P}(G1004)$ and $\text{rank}(A) = 4$.

Example 2.23.

$$msr(G1005) = 3$$

The tree size of $G1005$ is 4.

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & -2 & 1 \end{bmatrix}$$

Let

$$A = B^T B$$

Then $A \in \mathcal{P}(G1005)$ and $\text{rank}(A) = 3$.

Example 2.24.

$$msr(G1006) = 4$$

The tree size of $G1006$ is 4. If $v = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ is a representation of $G1006$, then $v_1 \perp \{v_3, v_4\}$, $v_2 \perp \{v_4, v_7\}$, $v_3 \perp \{v_5, v_6\}$, $v_4 \perp v_5$, $v_5 \perp v_7$, $v_6 \perp v_7$. We know $v_5 \perp \{v_3, v_4, v_7\}$. If $msr(G1006) = 3$, then there exists a, b, c such that $av_3 + bv_4 + cv_7 = 0$. From $av_3 \cdot v_1 + bv_4 \cdot v_1 + cv_7 \cdot v_1 = 0$, we get $c = 0$. From $av_3 \cdot v_6 + bv_4 \cdot v_6 = 0$, we get $b = 0$. Hence $av_3 = 0$ and it follows that $a = 0$. This is a contradiction.

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 \end{bmatrix}$$

Let

$$A = B^T B$$

Then $A \in \mathcal{P}(G1006)$ and $\text{rank}(A) = 4$.

Example 2.25.

$$msr(G1007) = 4$$

The tree size of $G1007$ is 5.

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 & 0 & -6 & 0 \\ 0 & 3 & 0 & 1 & 0 & 1 & 1 \\ 0 & 4 & 0 & 0 & 1 & 2 & -1 \end{bmatrix}$$

Let

$$A = B^T B$$

Then $A \in \mathcal{P}(G1007)$ and $\text{rank}(A) = 4$.

Example 2.26.

$$msr(G1053) = 4$$

The tree size of $G1053$ is 5.

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 & -3 \\ 0 & -2 & 1 & 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Let

$$A = B^T B$$

Then $A \in \mathcal{P}(G1053)$ and $\text{rank}(A) = 4$.

Example 2.27.

$$msr(G1056) = 3$$

The tree size of $G1056$ is 4.

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 2 & -1 \end{bmatrix}$$

Let

$$A = B^T B$$

Then $A \in \mathcal{P}(G1056)$ and $\text{rank}(A) = 3$.

Example 2.28.

$$\text{msr}(G1060) = 3$$

The tree size of $G1060$ is 4.

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Let

$$A = B^T B$$

Then $A \in \mathcal{P}(G1060)$ and $\text{rank}(A) = 3$.

Example 2.29.

$$\text{msr}(G1065) = 4$$

The tree size of $G1065$ is 4. If $v = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ is a representation of $G1065$, then $v_1 \perp \{v_6\}$, $v_2 \perp \{v_3, v_4, v_5, v_6\}$, $v_3 \perp \{v_4, v_7\}$, $v_5 \perp v_7$. If $\text{msr}(G1065) = 3$, then there exists a, b, c such that $v_5 = av_2 + bv_3 + cv_4$. From $av_2 \cdot v_2 + bv_3 \cdot v_2 + cv_4 \cdot v_2 = v_5 \cdot v_2 = 0$, we get $a = 0$. From $0 = v_5 \cdot v_7 = bv_3 \cdot v_7 + cv_4 \cdot v_7$, we get $c = 0$. Hence $v_5 = bv_3$. It follows that $v_5 \cdot v_4 = bv_3 \cdot v_4 = 0$. This is a contradiction.

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 2 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ -2 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & -1 & 1 & 1 \end{bmatrix}$$

Let

$$A = B^T B$$

Then $A \in \mathcal{P}(G1065)$ and $\text{rank}(A) = 4$.

Example 2.30.

$$\text{msr}(G1069) = 4$$

The tree size of $G1069$ is 4. If $v = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ is a representation of $G1069$, then $v_1 \perp \{v_3, v_4\}$, $v_2 \perp \{v_4\}$, $v_3 \perp \{v_7\}$, $v_4 \perp v_7$, $v_5 \perp \{v_6, v_7\}$, and $v_6 \perp v_7$. If $\text{msr}(G1069) = 3$, then $v_1 = kv_7$. It follows that $v_1 \cdot v_6 = kv_7 \cdot v_6 = 0$. This is a contradiction.

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & -3 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 3 & 1 & -2 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Let

$$A = B^T B$$

Then $A \in \mathcal{P}(G1069)$ and $\text{rank}(A) = 4$.

Example 2.31.

$$msr(G1084) = 4$$

The tree size of $G1084$ is 4. If $v = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ is a representation of $G1084$, then $v_1 \perp \{v_3, v_4, v_5\}$, $v_2 \perp \{v_5, \}$, $v_3 \perp \{v_4, v_7\}$, $v_4 \perp v_7$, and $v_6 \perp v_7$. If $msr(G1084) = 3$, then $v_1 = kv_7$. It follows that $v_1 \cdot v_6 = kv_7 \cdot v_6 = 0$. This is a contradiction.

Let

$$B = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Let

$$A = B^T B$$

Then $A \in \mathcal{P}(G1084)$ and $\text{rank}(A) = 4$.

Example 2.32.

$$msr(G1089) = 4$$

The tree size of $G1089$ is 5.

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & -4 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & -1 & -2 \\ 0 & -1 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

Let

$$A = B^T B$$

Then $A \in \mathcal{P}(G1089)$ and $\text{rank}(A) = 4$.

Example 2.33.

$$msr(G1090) = 3$$

The tree size of $G1090$ is 4.

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & -1 & 2 \\ 0 & -2 & 1 & 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 2 & 1 & 2 & 1 \end{bmatrix}$$

Let

$$A = B^T B$$

Then $A \in \mathcal{P}(G1090)$ and $\text{rank}(A) = 3$.

Example 2.34.

$$msr(G1091) = 4$$

The tree size of $G1091$ is 4. If $v = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ is a representation of $G1084$, then $v_1 \perp \{v_3, v_4, v_5\}$, $v_2 \perp \{v_4, v_5\}$, $v_3 \perp \{v_7\}$, $v_5 \perp v_7$, and $v_6 \perp v_7$. If $msr(G1091) = 3$, then $v_1 = kv_7$. It follows that $v_1 \cdot v_6 = kv_7 \cdot v_6 = 0$. This is a contradiction.

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 3 & -1 \end{bmatrix}$$

Let

$$A = B^T B$$

Then $A \in \mathcal{P}(G1091)$ and $\text{rank}(A) = 4$.

Example 2.35.

$$msr(G1092) = 4$$

The tree size of $G1092$ is 4. If $v = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ is a representation of $G1084$, then $v_1 \perp \{v_3, v_4\}$, $v_2 \perp \{v_4\}$, $v_3 \perp \{v_6, v_7\}$, $v_5 \perp \{v_6, v_7\}$, and $v_6 \perp v_7$. If $msr(G1092) = 3$, then $v_5 = kv_3$. It follows that $v_5 \cdot v_1 = kv_3 \cdot v_1 = 0$. This is a contradiction.

Let

$$B = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ -2 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & -3 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

Let

$$A = B^T B$$

Then $A \in \mathcal{P}(G1092)$ and $\text{rank}(A) = 4$.

Example 2.36.

$$msr(G1093) = 4$$

The tree size of $G1093$ is 5.

Let

$$B = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & 1 & 0 & -2 & 1 \end{bmatrix}$$

Let

$$A = B^T B$$

Then $A \in \mathcal{P}(G1093)$ and $\text{rank}(A) = 4$.

Example 2.37.

$$msr(G1094) = 3$$

The tree size of $G1094$ is 4.

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & -1 & -1 \end{bmatrix}$$

Let

$$A = B^T B$$

Then $A \in \mathcal{P}(G1094)$ and $\text{rank}(A) = 3$.

Example 2.38.

$$msr(G1095) = 3$$

The tree size of $G1095$ is 4.

Let

$$B = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ -1 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Let

$$A = B^T B$$

Then $A \in \mathcal{P}(G1095)$ and $\text{rank}(A) = 3$.

Example 2.39.

$$msr(G1096) = 3$$

The tree size of $G1096$ is 4.

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & -3 & 0 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 & -1 & 1 & -1 \end{bmatrix}$$

Let

$$A = B^T B$$

Then $A \in \mathcal{P}(G1096)$ and $\text{rank}(A) = 3$.

Example 2.40.

$$msr(G1097) = 4$$

The tree size of $G1097$ is 4. If $v = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ is a representation of $G1097$, then $v_1 \perp \{v_3, v_4\}$, $v_2 \perp \{v_4, v_5, v_6\}$, $v_3 \perp \{v_5, v_6\}$, and $v_5 \perp \{v_7\}$. If $msr(G1097) = 3$, then $v_2 = kv_3$. It follows that $v_2 \cdot v_1 = kv_3 \cdot v_1 = 0$. This is a contradiction.

Let

$$B = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & -1 \\ 1 & 0 & -1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Let

$$A = B^T B$$

Then $A \in \mathcal{P}(G1097)$ and $\text{rank}(A) = 4$.

Example 2.41.

$$msr(G1100) = 3$$

The tree size of $G1100$ is 4.

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 1 & 1 & 3 & 0 \\ 0 & 2 & 1 & 0 & -1 & 1 & 1 \end{bmatrix}$$

Let

$$A = B^T B$$

Then $A \in \mathcal{P}(G1100)$ and $\text{rank}(A) = 3$.

Example 2.42.

$$msr(G1101) = 4$$

The tree size of $G1101$ is 4. If $v = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ is a representation of $G1101$, then $v_1 \perp \{v_3, v_4, v_6\}$, $v_2 \perp \{v_4, v_7\}$, $v_3 \perp \{v_7\}$, $v_4 \perp v_5$, and $v_5 \perp \{v_6\}$. If $msr(G1101) = 3$, then $v_5 = kv_1$. It follows that $v_5 \cdot v_3 = kv_3 \cdot v_1 = 0$. This is a contradiction.

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 & -1 & 1 & 2 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Let

$$A = B^T B$$

Then $A \in \mathcal{P}(G1101)$ and $\text{rank}(A) = 4$.

Example 2.43.

$$msr(G1104) = 3$$

The tree size of $G1104$ is 4.

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 & -2 \\ 0 & 1 & 1 & 1 & 0 & 0 & 3 \\ 0 & -1 & 0 & 1 & 2 & 1 & 1 \end{bmatrix}$$

Let

$$A = B^T B$$

Then $A \in \mathcal{P}(G1104)$ and $\text{rank}(A) = 3$.

Example 2.44.

$$msr(G1105) = 3$$

The tree size of $G1105$ is 4.

Let

$$B = \begin{bmatrix} 5 & 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & -3 & 1 & 1 & 1 & 0 \\ 4 & 0 & -2 & -1 & 0 & 1 & 1 \end{bmatrix}$$

Let

$$A = B^T B$$

Then $A \in \mathcal{P}(G1105)$ and $\text{rank}(A) = 3$.

Example 2.45.

$$msr(G1123) = 3$$

The tree size of $G1123$ is 4.

Let

$$B = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 2 & 0 & 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

Let

$$A = B^T B$$

Then $A \in \mathcal{P}(G1123)$ and $\text{rank}(A) = 3$.

Example 2.46.

$$msr(G1125) = 3$$

The tree size of $G1125$ is 4.

Let

$$B = \begin{bmatrix} 1 & -2 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 2 \end{bmatrix}$$

Let

$$A = B^T B$$

Then $A \in \mathcal{P}(G1125)$ and $\text{rank}(A) = 3$.

Example 2.47.

$$msr(G1135) = 3$$

The tree size of $G1135$ is 4.

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & -2 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 2 & -1 \end{bmatrix}$$

Let

$$A = B^T B$$

Then $A \in \mathcal{P}(G1135)$ and $\text{rank}(A) = 3$.

Example 2.48.

$$msr(G1145) = 4$$

The tree size of $G1145$ is 4. If $v = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ is a representation of $G1145$, then $v_1 \perp \{v_3, v_4, v_5\}$, $v_2 \perp \{v_4, v_5\}$, $v_4 \perp \{v_6\}$, and $v_5 \perp v_7$. If $msr(G1145) = 3$, then $v_2 = kv_1$. It follows that $v_2 \cdot v_3 = kv_3 \cdot v_1 = 0$. This is a contradiction.

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Let

$$A = B^T B$$

Then $A \in \mathcal{P}(G1145)$ and $\text{rank}(A) = 4$.

Example 2.49.

$$msr(G1146) = 3$$

The tree size of $G1146$ is 4.

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 2 & 0 \end{bmatrix}$$

Let

$$A = B^T B$$

Then $A \in \mathcal{P}(G1146)$ and $\text{rank}(A) = 3$.

Example 2.50.

$$msr(G1148) = 4$$

The tree size of $G1148$ is 5. Let

$$A = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 & 0 & 1 & 1 \\ -1 & -1 & 3 & -1 & 0 & -1 & -3 \\ 0 & 0 & -1 & 2 & -1 & -2 & 2 \\ 0 & 0 & 0 & -1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -2 & 1 & 4 & 0 \\ 1 & 1 & -3 & 2 & -1 & 0 & 4 \end{bmatrix}$$

Let

$$B = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & -1 & -1 & 1 \end{bmatrix}$$

Then

$$A = B^T B$$

$A \in \mathcal{P}(G1148)$ and $\text{rank}(A) = 4$.

Example 2.51.

$$msr(G1149) = 3$$

The tree size of $G1149$ is 4. Let

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 & 1 & 1 & 0 \\ -1 & 2 & -1 & 0 & -2 & 0 & -1 \\ 0 & -1 & 2 & -1 & 2 & 0 & 1 \\ 0 & 0 & -1 & 1 & -1 & -1 & 0 \\ 1 & -2 & 2 & -1 & 3 & 1 & 1 \\ 1 & 0 & 0 & -1 & 1 & 3 & -1 \\ 0 & -1 & 1 & 0 & 1 & -1 & 1 \end{bmatrix}$$

Let

$$B = \begin{bmatrix} 1 & -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & -1 & 1 & 1 & 0 \end{bmatrix}$$

Then

$$A = B^T B$$

$A \in \mathcal{P}(G1149)$ and $\text{rank}(A) = 3$.

Example 2.52.

$$msr(G1152) = 3$$

The tree size of $G1152$ is 4. Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 & -2 \\ 1 & 3 & -1 & 0 & 2 & 1 & 0 \\ 0 & -1 & 1 & 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 2 & 5 & 1 & 0 \\ 1 & 2 & 2 & 5 & 14 & 3 & -1 \\ 0 & 1 & 0 & 1 & 3 & 1 & 1 \\ -2 & 0 & -1 & 0 & -1 & 1 & 6 \end{bmatrix}$$

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 & -2 \\ 0 & 1 & -1 & -1 & -2 & 0 & 1 \\ 0 & 1 & 0 & 1 & 3 & 1 & 1 \end{bmatrix}$$

Then

$$A = B^T B$$

$A \in \mathcal{P}(G1152)$ and $\text{rank}(A) = 3$.

Example 2.53.

$$msr(G1153) = 3$$

The tree size of $G1153$ is 4. Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & -1 & -1 & 1 \\ 1 & 2 & -1 & 0 & -4 & -4 & 0 \\ 0 & -1 & 5 & 2 & 7 & 7 & -1 \\ 0 & 0 & 2 & 1 & 2 & 2 & -1 \\ -1 & -4 & 7 & 2 & 14 & 7 & 0 \\ -1 & -4 & 7 & 2 & 7 & 14 & 0 \\ 1 & 0 & -1 & -1 & 0 & 0 & 3 \end{bmatrix}$$

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & -1 & -1 & 1 \\ 0 & 1 & -1 & 0 & -3 & -3 & -1 \\ 0 & 0 & 2 & 1 & 2 & 2 & -1 \end{bmatrix}$$

Then

$$A = B^T B$$

$A \in \mathcal{P}(G1153)$ and $\text{rank}(A) = 3$.

Example 2.54.

$$msr(G1154) = 4$$

The tree size of $G1154$ is 4. First let's show $msr(G1154) > 3$.

Let $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ be a vector representation of $G1154$. It follows that $v_1 \perp v_3$, $v_1 \perp v_4$, $v_1 \perp v_7$, $v_3 \perp v_6$, and $v_4 \perp v_6$. If $msr(G1154) = 3$, then $v_7 \in \text{span}\{v_3, v_4\}$. It follows that $v_6 \perp v_7$. We know that $v_6 \cdot v_7 \neq 0$. This leads a contradiction.

On the other hand, let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 2 & 1 & 0 & 3 & 1 & 1 \\ 0 & 1 & 2 & 1 & 5 & 0 & 5 \\ 0 & 0 & 1 & 1 & 3 & 0 & 4 \\ 1 & 3 & 5 & 3 & 18 & 3 & 0 \\ 1 & 1 & 0 & 0 & 3 & 2 & -7 \\ 0 & 1 & 5 & 4 & 0 & -7 & 66 \end{bmatrix}$$

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 & 3 & 0 & 4 \\ 0 & 0 & 0 & 0 & 2 & 1 & -7 \end{bmatrix}$$

Then

$$A = B^T B$$

$A \in \mathcal{P}(G1154)$ and $\text{rank}(A) = 4$.

Example 2.55.

$$msr(G1155) = 3$$

The tree size of $G1155$ is 4. Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & -1 & 0 & 1 \\ 1 & 2 & 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 2 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & -1 \\ -1 & 0 & 2 & 1 & 3 & 1 & -1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 2 & 0 & -1 & -1 & 1 & 3 \end{bmatrix}$$

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & -1 \end{bmatrix}$$

Then

$$A = B^T B$$

$A \in \mathcal{P}(G1155)$ and $\text{rank}(A) = 3$.

Example 2.56.

$$msr(G1156) = 3$$

The tree size of $G1156$ is 4. Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 3 & -1 \\ 0 & 1 & 2 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 2 \\ 1 & 0 & 0 & 1 & 3 & -1 & 3 \\ 1 & 3 & 2 & 0 & -1 & 5 & -2 \\ 0 & -1 & 1 & 2 & 3 & -2 & 5 \end{bmatrix}$$

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 2 \end{bmatrix}$$

Then

$$A = B^T B$$

$A \in \mathcal{P}(G1156)$ and $\text{rank}(A) = 3$.

Example 2.57.

$$msr(G1157) = 3$$

The tree size of $G1157$ is 4. Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 & -5 \\ 1 & 2 & 1 & 0 & 3 & 1 & -3 \\ 0 & 1 & 2 & 1 & 3 & -1 & 3 \\ 0 & 0 & 1 & 1 & 1 & -2 & 1 \\ 1 & 3 & 3 & 1 & 6 & 0 & 0 \\ 0 & 1 & -1 & -2 & 0 & 5 & 0 \\ -5 & -3 & 3 & 1 & 0 & 0 & 30 \end{bmatrix}$$

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 & -5 \\ 0 & 1 & 1 & 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 & -2 & 1 \end{bmatrix}$$

Then

$$A = B^T B$$

$A \in \mathcal{P}(G1157)$ and $\text{rank}(A) = 3$.

Example 2.58.

$$msr(G1159) = 3$$

The tree size of $G1159$ is 4. Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 2 & 1 & 2 \\ 1 & 5 & 2 & 0 & 0 & -1 & 0 \\ 0 & 2 & 2 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 2 \\ 2 & 0 & 0 & 1 & 6 & 4 & 7 \\ 1 & -1 & 0 & 1 & 4 & 3 & 5 \\ 2 & 0 & 1 & 2 & 7 & 5 & 9 \end{bmatrix}$$

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 2 & 1 & 2 \\ 0 & 2 & 1 & 0 & -1 & -1 & -1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 2 \end{bmatrix}$$

Then

$$A = B^T B$$

$A \in \mathcal{P}(G1159)$ and $\text{rank}(A) = 3$.

Example 2.59.

$$msr(G1160) = 4$$

The tree size of $G1160$ is 5. Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 2 & 1 \\ 1 & 2 & 1 & 0 & 0 & 13 & -1 \\ 0 & 1 & 3 & 3 & 0 & 7 & -4 \\ 0 & 0 & 3 & 5 & 1 & -2 & -1 \\ 0 & 0 & 0 & 1 & 2 & 8 & 4 \\ 2 & 13 & 7 & -2 & 8 & 165 & 0 \\ 1 & -1 & -4 & -1 & 4 & 0 & 15 \end{bmatrix}$$

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 & 0 & 11 & -2 \\ 0 & 0 & 1 & 2 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 & -1 & -6 & -3 \end{bmatrix}$$

Then

$$A = B^T B$$

$A \in \mathcal{P}(G1160)$ and $\text{rank}(A) = 4$.

Example 2.60.

$$msr(G1165) = 3$$

The tree size of $G1165$ is 4. Let

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 & 2 & -1 & 1 \\ 2 & 6 & -1 & -1 & 3 & -3 & 0 \\ 0 & -1 & 1 & 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 & 1 & 2 & 1 \\ 2 & 3 & 0 & 1 & 5 & 0 & 3 \\ -1 & -3 & -1 & 2 & 0 & 6 & 0 \\ 1 & 0 & 1 & 1 & 3 & 0 & 3 \end{bmatrix}$$

Let

$$B = \begin{bmatrix} 1 & 2 & 0 & 0 & 2 & -1 & 1 \\ 0 & 1 & -1 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 & -1 & -2 & -1 \end{bmatrix}$$

Then

$$A = B^T B$$

$A \in \mathcal{P}(G1165)$ and $\text{rank}(A) = 3$.

Example 2.61.

$$msr(G1167) = 3$$

The tree size of $G1167$ is 4. Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 3 & 1 & 1 \\ 1 & 6 & -1 & 0 & 0 & 8 & 0 \\ 0 & -1 & 2 & 3 & 0 & -2 & -1 \\ 0 & 0 & 3 & 5 & -1 & -1 & -2 \\ 3 & 0 & 0 & -1 & 11 & -1 & 4 \\ 1 & 8 & -2 & -1 & -1 & 11 & 0 \\ 1 & 0 & -1 & -2 & 4 & 0 & 2 \end{bmatrix}$$

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 3 & 1 & 1 \\ 0 & 2 & -1 & -1 & -1 & 3 & 0 \\ 0 & 1 & 1 & 2 & -1 & 1 & -1 \end{bmatrix}$$

Then

$$A = B^T B$$

$A \in \mathcal{P}(G1167)$ and $\text{rank}(A) = 3$.

Example 2.62.

$$msr(G1168) = 3$$

The tree size of $G1168$ is 4. Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 6 & -1 & 0 & 4 & 0 & -4 \\ 0 & -1 & 2 & 3 & 0 & -1 & 5 \\ 0 & 0 & 3 & 5 & 1 & -2 & 7 \\ 1 & 4 & 0 & 1 & 3 & 0 & -1 \\ 1 & 0 & -1 & -2 & 0 & 2 & -2 \\ 0 & -4 & 5 & 7 & -1 & -2 & 13 \end{bmatrix}$$

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 1 & -1 & 2 \\ 0 & 2 & -1 & -1 & 1 & 0 & -3 \end{bmatrix}$$

Then

$$A = B^T B$$

$A \in \mathcal{P}(G1168)$ and $\text{rank}(A) = 3$.

Example 2.63.

$$msr(G1170) = 3$$

The tree size of $G1170$ is 4. Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & -1 & -2 & -2 \\ 1 & 2 & 1 & 0 & 0 & -3 & -1 \\ 0 & 1 & 2 & 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 1 & 1 & 1 & -3 \\ -1 & 0 & 2 & 1 & 3 & 2 & 0 \\ -2 & -3 & 0 & 1 & 2 & 6 & 0 \\ -2 & -1 & -2 & -3 & 0 & 0 & 14 \end{bmatrix}$$

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & -1 & -2 & -2 \\ 0 & 1 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & -3 \end{bmatrix}$$

Then

$$A = B^T B$$

$A \in \mathcal{P}(G1170)$ and $\text{rank}(A) = 3$.

Example 2.64.

$$\text{msr}(G1176) = 3$$

The tree size of $G1176$ is 4.

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 & 1 & 1 & -1 \end{bmatrix}$$

Let

$$A = B^T B$$

$A \in \mathcal{P}(G1176)$ and $\text{rank}(A) = 3$.

Example 2.65.

$$\text{msr}(G1179) = 3$$

The tree size of $G1179$ is 4. Let

$$A = \begin{bmatrix} 1 & 5 & 0 & 0 & 1 & 3 & 6 \\ 5 & 45 & 2 & 0 & 7 & 5 & 0 \\ 0 & 2 & 2 & 3 & 2 & -1 & -3 \\ 0 & 0 & 3 & 5 & 3 & 0 & 0 \\ 1 & 7 & 2 & 3 & 3 & 2 & 3 \\ 3 & 5 & -1 & 0 & 2 & 14 & 33 \\ 6 & 0 & -3 & 0 & 3 & 33 & 81 \end{bmatrix}$$

Let

$$B = \begin{bmatrix} 1 & 5 & 0 & 0 & 1 & 3 & 6 \\ 0 & 2 & -1 & -2 & -1 & -1 & -3 \\ 0 & 4 & 1 & 1 & 1 & -2 & -6 \end{bmatrix}$$

Then

$$A = B^T B$$

$A \in \mathcal{P}(G1179)$ and $\text{rank}(A) = 3$.

Example 2.66.

$$msr(G1189) = 3$$

The tree size of $G1189$ is 4. Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 6 & 3 & 1 & 2 & 7 & 5 \\ 0 & 3 & 2 & 0 & 0 & 5 & 3 \\ 0 & 1 & 0 & 2 & 2 & -1 & -1 \\ 1 & 2 & 0 & 2 & 3 & -1 & 0 \\ 0 & 7 & 5 & -1 & -1 & 13 & 8 \\ 1 & 5 & 3 & -1 & 0 & 8 & 6 \end{bmatrix}$$

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 2 & 1 & 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & -1 & -1 & 3 & 2 \end{bmatrix}$$

Then

$$A = B^T B$$

$A \in \mathcal{P}(G1189)$ and $\text{rank}(A) = 3$.

Example 2.67.

$$msr(G1191) = 3$$

The tree size of $G1191$ is 4. Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 14 & -1 & 5 & 3 & 8 & 2 \\ 0 & -1 & 2 & 0 & 1 & 1 & -2 \\ 0 & 5 & 0 & 2 & 1 & 3 & 0 \\ 1 & 3 & 1 & 1 & 2 & 3 & 0 \\ 1 & 8 & 1 & 3 & 3 & 6 & 0 \\ 1 & 2 & -2 & 0 & 0 & 0 & 3 \end{bmatrix}$$

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 & 1 & 2 & -1 \\ 0 & 3 & -1 & 1 & 0 & 1 & 1 \end{bmatrix}$$

Then

$$A = B^T B$$

$A \in \mathcal{P}(G1191)$ and $\text{rank}(A) = 3$.

Example 2.68.

$$msr(G1194) = 3$$

The tree size of $G1194$ is 4. Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 6 & -1 & 0 & -2 & -1 & 6 \\ 0 & -1 & 2 & 3 & 0 & 5 & -1 \\ 0 & 0 & 3 & 5 & -1 & 8 & 0 \\ 1 & -2 & 0 & -1 & 3 & -1 & -2 \\ 0 & -1 & 5 & 8 & -1 & 13 & -1 \\ 1 & 6 & -1 & 0 & -2 & -1 & 6 \end{bmatrix}$$

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 2 & -1 & -1 & -1 & -2 & 2 \\ 0 & 1 & 1 & 2 & -1 & 3 & 1 \end{bmatrix}$$

Then

$$A = B^T B$$

$A \in \mathcal{P}(G1194)$ and $\text{rank}(A) = 3$.

Example 2.69.

$$msr(G1195) = 3$$

The tree size of $G1195$ is 4. Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 4 & 0 & 2 \\ 1 & 3 & 1 & 0 & 5 & 2 & 6 \\ 0 & 1 & 5 & 3 & 11 & 1 & -6 \\ 0 & 0 & 3 & 2 & 7 & 0 & -1 \\ 4 & 5 & 11 & 7 & 41 & 1 & 0 \\ 0 & 2 & 1 & 0 & 1 & 2 & -9 \\ 2 & -7 & -6 & -1 & 0 & -9 & 45 \end{bmatrix}$$

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 4 & 0 & 2 \\ 0 & 1 & 2 & 1 & 4 & 1 & -5 \\ 0 & 1 & -1 & -1 & -3 & 1 & -4 \end{bmatrix}$$

Then

$$A = B^T B$$

$A \in \mathcal{P}(G1195)$ and $\text{rank}(A) = 3$.

Example 2.70.

$$msr(G1196) = 3$$

The tree size of $G1196$ is 4. Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 2 & 2 \\ 1 & 19 & 9 & 0 & -2 & -1 & 8 \\ 0 & 9 & 5 & -1 & 0 & 0 & 3 \\ 0 & 0 & -1 & 2 & -3 & -3 & 0 \\ 1 & -2 & 0 & -3 & 5 & 7 & 1 \\ 2 & -1 & 0 & -3 & 7 & 9 & 3 \\ 2 & 8 & 3 & 0 & 1 & 3 & 6 \end{bmatrix}$$

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 2 & 2 \\ 0 & 3 & 1 & 1 & -2 & -2 & 1 \\ 0 & 3 & 2 & -1 & 1 & 1 & 1 \end{bmatrix}$$

Then

$$A = B^T B$$

$A \in \mathcal{P}(G1196)$ and $\text{rank}(A) = 3$.

Example 2.71.

$$msr(G1197) = 3$$

The tree size of $G1197$ is 4. Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 & 2 \\ 1 & 2 & -1 & -2 & -2 & 0 & 0 \\ 0 & -1 & 5 & 0 & 7 & -1 & 0 \\ 0 & -2 & 0 & 5 & 4 & 3 & 5 \\ 1 & -2 & 7 & 4 & 14 & 2 & 6 \\ 1 & 0 & -1 & 3 & 2 & 3 & 5 \\ 2 & 0 & 0 & 5 & 6 & 5 & 9 \end{bmatrix}$$

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 & 2 \\ 0 & -1 & 1 & 2 & 3 & 1 & 2 \\ 0 & 0 & 2 & -1 & 2 & -1 & -1 \end{bmatrix}$$

Then $B^T B = A$, $A \in \mathcal{P}(G1197)$ and $\text{rank}(A) = 3$.

Example 2.72.

$$msr(G1199) = 3$$

The tree size of $G1199$ is 4. Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 2 & 0 \\ 1 & 14 & 5 & 13 & 22 & 9 & 5 \\ 0 & 5 & 2 & 5 & 9 & 1 & 1 \\ 1 & 13 & 5 & 14 & 23 & 0 & 0 \\ 0 & 22 & 9 & 23 & 41 & 0 & 2 \\ 2 & 9 & 1 & 0 & 0 & 45 & 23 \\ 0 & 5 & 1 & 0 & 2 & 23 & 13 \end{bmatrix}$$

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & 2 & 1 & 3 & 5 & -4 & -2 \\ 0 & 3 & 1 & 2 & 4 & 5 & 3 \end{bmatrix}$$

Then $B^T B = A$, $A \in \mathcal{P}(G1199)$ and $\text{rank}(A) = 3$.

Example 2.73.

$$msr(G1200) = 3$$

The tree size of $G1200$ is 4. Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 3 & -1 & 0 & 6 & 1 & 1 \\ 0 & -1 & 5 & 3 & -4 & 7 & 3 \\ 0 & 0 & 3 & 2 & -1 & 5 & 2 \\ 1 & 6 & -4 & -1 & 14 & 0 & 0 \\ 0 & 1 & 7 & 5 & 0 & 13 & 5 \\ 1 & 1 & 3 & 2 & 0 & 5 & 3 \end{bmatrix}$$

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & -1 & 2 & 1 & -3 & 2 & 1 \\ 0 & 1 & 1 & 1 & 2 & 3 & 1 \end{bmatrix}$$

Then $B^T B = A$, $A \in \mathcal{P}(G1200)$ and $\text{rank}(A) = 3$.

Example 2.74.

$$msr(G1202) = 3$$

The tree size of $G1202$ is 4. Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 2 \\ 1 & 6 & -1 & 3 & 0 & 9 & -1 \\ 0 & -1 & 2 & 0 & 3 & -1 & 9 \\ 0 & 3 & 0 & 2 & 1 & 5 & 1 \\ 0 & 0 & 3 & 1 & 5 & 1 & 14 \\ 1 & 9 & -1 & 5 & 1 & 14 & 0 \\ 2 & -1 & 9 & 1 & 14 & 0 & 45 \end{bmatrix}$$

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 2 \\ 0 & -2 & 1 & -1 & 1 & -3 & 4 \\ 0 & 1 & 1 & 1 & 2 & 2 & 5 \end{bmatrix}$$

Then $B^T B = A$, $A \in \mathcal{P}(G1202)$ and $\text{rank}(A) = 3$.

Example 2.75.

$$msr(G1205) = 3$$

The tree size of $G1205$ is 4. Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & -1 & 2 \\ 1 & 6 & 3 & 0 & -1 & 0 & 7 \\ 0 & 3 & 5 & 4 & 2 & 3 & 3 \\ 0 & 0 & 4 & 5 & 4 & 3 & 0 \\ 1 & -1 & 2 & 4 & 5 & 1 & 0 \\ -1 & 0 & 3 & 3 & 1 & 3 & -1 \\ 2 & 7 & 3 & 0 & 0 & -1 & 9 \end{bmatrix}$$

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & -1 & 2 \\ 0 & -1 & 1 & 2 & 2 & 1 & -1 \\ 0 & 2 & 2 & 1 & 0 & 1 & 2 \end{bmatrix}$$

Then $B^T B = A$, $A \in \mathcal{P}(G1205)$ and $\text{rank}(A) = 3$.

Example 2.76.

$$msr(G1207) = 3$$

The tree size of $G1207$ is 4. Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & -1 & 1 & 1 \\ 1 & 6 & 3 & 0 & 0 & 5 & -3 \\ 0 & 3 & 5 & 4 & 3 & 8 & 0 \\ 0 & 0 & 4 & 5 & 3 & 7 & 3 \\ -1 & 0 & 3 & 3 & 3 & 4 & 0 \\ 1 & 5 & 8 & 7 & 4 & 14 & 2 \\ 1 & -3 & 0 & 3 & 0 & 2 & 6 \end{bmatrix}$$

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & -1 & 1 & 1 \\ 0 & -1 & 1 & 2 & 1 & 2 & 2 \\ 0 & 2 & 2 & 1 & 1 & 3 & -1 \end{bmatrix}$$

Then $B^T B = A$, $A \in \mathcal{P}(G1207)$ and $\text{rank}(A) = 3$.

Example 2.77.

$$msr(G1208) = 3$$

The tree size of $G1208$ is 4. Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 3 & 3 & 1 & 0 & 0 & 3 \\ 0 & 3 & 5 & 0 & -1 & -3 & 4 \\ 0 & 1 & 0 & 5 & -2 & 4 & -2 \\ 1 & 0 & -1 & -2 & 2 & 0 & 1 \\ 1 & 0 & -3 & 4 & 0 & 6 & -3 \\ 1 & 3 & 4 & -2 & 1 & -3 & 5 \end{bmatrix}$$

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & -1 & 0 & -2 & 2 \\ 0 & 1 & 1 & 2 & -1 & 1 & 0 \end{bmatrix}$$

Then $B^T B = A$, $A \in \mathcal{P}(G1208)$ and $\text{rank}(A) = 3$.

Example 2.78.

$$msr(G1209) = 3$$

The tree size of $G1209$ is 4. Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 2 & 1 & 1 \\ 1 & 6 & 3 & 0 & 1 & 7 & 5 \\ 0 & 3 & 5 & 4 & 1 & 2 & 0 \\ 0 & 0 & 4 & 5 & 2 & -2 & -3 \\ 2 & 1 & 1 & 2 & 5 & 0 & 0 \\ 1 & 7 & 2 & -2 & 0 & 9 & 7 \\ 1 & 5 & 0 & -3 & 0 & 7 & 6 \end{bmatrix}$$

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 2 & 1 & 1 \\ 0 & -1 & 1 & 2 & 1 & -2 & -2 \\ 0 & 2 & 2 & 1 & 0 & 2 & 1 \end{bmatrix}$$

Then $B^T B = A$, $A \in \mathcal{P}(G1209)$ and $\text{rank}(A) = 3$.

Example 2.79.

$$msr(G1210) = 3$$

The tree size of $G1210$ is 4. Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 3 & 1 \\ 1 & 3 & -1 & 0 & 2 & 2 & 9 \\ 0 & -1 & 5 & 3 & 7 & -1 & -7 \\ 0 & 0 & 3 & 2 & 5 & -1 & -2 \\ 1 & 2 & 7 & 5 & 14 & 0 & -1 \\ 3 & 2 & -1 & -1 & 0 & 10 & 0 \\ 1 & 9 & -7 & -2 & -1 & 0 & 35 \end{bmatrix}$$

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 3 & 1 \\ 0 & -1 & 2 & 1 & 2 & 0 & -5 \\ 0 & 1 & 1 & 1 & 3 & -1 & 3 \end{bmatrix}$$

Then $B^T B = A$, $A \in \mathcal{P}(G1210)$ and $\text{rank}(A) = 3$.

Example 2.80.

$$msr(G1211) = 2$$

The tree size of $G1211$ is 3. Let

$$A = \begin{bmatrix} 1 & 1 & 2 & -1 & -1 & -2 & 0 \\ 1 & 5 & 0 & -1 & -5 & 0 & 2 \\ 2 & 0 & 5 & -2 & 0 & -5 & -1 \\ -1 & -1 & -2 & 1 & 1 & 2 & 0 \\ -1 & -5 & 0 & 1 & 5 & 0 & -2 \\ -2 & 0 & -5 & 2 & 0 & 5 & 1 \\ 0 & 2 & -1 & 0 & -2 & 1 & 1 \end{bmatrix}$$

Let

$$B = \begin{bmatrix} 1 & 1 & 2 & -1 & -1 & -2 & 0 \\ 0 & 2 & -1 & 0 & -2 & 1 & 1 \end{bmatrix}$$

Then $B^T B = A$, $A \in \mathcal{P}(G1211)$ and $\text{rank}(A) = 2$.

Example 2.81.

$$msr(G1212) = 3$$

The tree size of $G1212$ is 4. Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 3 & 1 & 2 \\ 1 & 3 & 1 & 0 & 0 & 2 & 3 \\ 0 & 1 & 5 & 3 & 0 & 8 & -1 \\ 0 & 0 & 3 & 2 & 1 & 5 & -1 \\ 3 & 0 & 0 & 1 & 10 & 4 & 4 \\ 1 & 2 & 8 & 5 & 4 & 14 & 0 \\ 2 & 3 & -1 & -1 & 4 & 0 & 5 \end{bmatrix}$$

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 3 & 1 & 2 \\ 0 & -1 & 1 & 1 & 2 & 2 & -1 \\ 0 & 1 & 2 & 1 & -1 & 3 & 0 \end{bmatrix}$$

Then $B^T B = A$, $A \in \mathcal{P}(G1212)$ and $\text{rank}(A) = 3$.

Example 2.82.

$$msr(G1222) = 3$$

The tree size of $G1222$ is 3. First let's show $msr(G1222) > 2$.

If $v = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ is a representation of $G1222$, It follows that $v_1 \perp v_3$, $v_3 \perp v_4$, $v_4 \perp v_6$. If $msr(G1222) = 2$, then $v_1 \perp v_6$. We know that $v_1 \cdot v_6 \neq 0$. This leads a contradiction.

On the other hand, let

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 3 & 2 & 1 & 6 & 2 & 1 \\ 0 & 2 & 2 & 0 & 5 & 1 & 0 \\ 1 & 1 & 0 & 3 & 2 & 0 & -1 \\ 1 & 6 & 5 & 2 & 14 & 3 & 0 \\ 1 & 2 & 1 & 0 & 3 & 2 & 2 \\ 1 & 1 & 0 & -1 & 0 & 2 & 3 \end{bmatrix}$$

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & -1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 & 3 & 0 & -1 \end{bmatrix}$$

Then $B^T B = A$, $A \in \mathcal{P}(G1222)$ and $\text{rank}(A) = 3$.

Example 2.83.

$$msr(G1224) = 3$$

The tree size of $G1224$ is 3. First let's show $msr(G1224) > 2$.

If $v = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ is a representation of $G1224$, It follows that $v_1 \perp v_5$, $v_1 \perp v_6$. If $msr(G1224) = 2$, then $v_5 \cdot v_6 \neq 0$. We know that $v_5 \cdot v_6 = 0$. This leads a contradiction.

On the other hand, let

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 5 & 2 & 3 & 2 & -4 & -1 \\ 0 & 2 & 2 & 2 & 3 & -1 & -1 \\ 1 & 3 & 2 & 3 & 3 & -1 & 0 \\ 0 & 2 & 3 & 3 & 5 & 0 & -1 \\ 0 & -4 & -1 & -1 & 0 & 5 & 2 \\ 1 & -1 & -1 & 0 & -1 & 2 & 2 \end{bmatrix}$$

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 2 & 1 & 1 & 1 & -2 & -1 \\ 0 & 0 & 1 & 1 & 2 & 1 & 0 \end{bmatrix}$$

Then $B^T B = A$, $A \in \mathcal{P}(G1224)$ and $\text{rank}(A) = 3$.

Example 2.84.

$$msr(G1228) = 3$$

The tree size of $G1228$ is 3. First let's show $msr(G1228) > 2$.

Let $v = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ be a representation of $G1228$. It follows that $v_1 \perp v_3$, $v_3 \perp v_4$, $v_4 \perp v_7$. If $msr(G1228) = 2$, then $v_1 \perp v_7$. We know that $v_1 \cdot v_7 \neq 0$. This leads a contradiction.

On the other hand, let

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 5 & 2 & -1 & 2 & 5 & 3 \\ 0 & 2 & 2 & 2 & 3 & -1 & -1 \\ 1 & 3 & 2 & 0 & 3 & 1 & 1 \\ 1 & -1 & 0 & 3 & 1 & -2 & 0 \\ 1 & 5 & 1 & -2 & 0 & 6 & 3 \\ 1 & 3 & 1 & 0 & 1 & 3 & 2 \end{bmatrix}$$

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & -1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 & 2 & -1 & 0 \end{bmatrix}$$

Then $B^T B = A$, $A \in \mathcal{P}(G1228)$ and $\text{rank}(A) = 3$.

Example 2.85.

$$msr(G1230) = 2$$

The tree size of $G1230$ is 3. Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 & -1 & 1 & -1 \\ 1 & 5 & 2 & 3 & -3 & -1 & 1 \\ 0 & 2 & 1 & 1 & -1 & -1 & 1 \\ 1 & 3 & 1 & 2 & -2 & 0 & 0 \\ -1 & -3 & -1 & -2 & 2 & 0 & 0 \\ 1 & -1 & -1 & 0 & 0 & 2 & -2 \\ -1 & 1 & 1 & 0 & 0 & -2 & 2 \end{bmatrix}$$

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 1 & -1 & 1 & -1 \\ 0 & 2 & 1 & 1 & -1 & -1 & 1 \end{bmatrix}$$

Then $B^T B = A$, $A \in \mathcal{P}(G1230)$ and $\text{rank}(A) = 2$.

Example 2.86.

$$msr(G1231) = 3$$

The tree size of $G1231$ is 3. First let's show $msr(G1231) > 2$.

Let $v = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ be a representation of $G1231$. It follows that $v_2 \perp v_4$, $v_2 \perp v_7$, $v_5 \perp v_7$. If $msr(G1231) = 2$, then $v_4 \perp v_5$. We know that $v_4 \cdot v_5 \neq 0$. This leads a contradiction.

On the other hand, let

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 0 & 4 & 2 & 0 \\ 0 & 1 & 5 & -1 & 7 & 3 & 1 \\ 1 & 0 & -1 & 2 & -2 & 0 & 2 \\ 1 & 4 & 7 & -2 & 14 & 6 & 0 \\ 1 & 2 & 3 & 0 & 6 & 3 & 1 \\ 1 & 0 & 1 & 2 & 0 & 1 & 3 \end{bmatrix}$$

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & -1 & 3 & 1 & -1 \\ 0 & 0 & 2 & 0 & 2 & 1 & 1 \end{bmatrix}$$

Then $B^T B = A$, $A \in \mathcal{P}(G1231)$ and $\text{rank}(A) = 3$.

Example 2.87.

$$msr(G1233) = 3$$

The tree size of $G1233$ is 3. First let's show $msr(G1233) > 2$.

Let $v = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ be a representation of $G1233$. It follows that $v_2 \perp v_4$, $v_2 \perp v_6$. If $msr(G1233) = 2$, then $v_4 \cdot v_6 \neq 0$. We know that $v_4 \cdot v_6 = 0$. This leads a contradiction.

On the other hand, let

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & -1 & -3 \\ 1 & 2 & 1 & 0 & 1 & 0 & -3 \\ 0 & 1 & 5 & 1 & 6 & 5 & 2 \\ 1 & 0 & 1 & 3 & 4 & 0 & -2 \\ 1 & 1 & 6 & 4 & 10 & 5 & 0 \\ -1 & 0 & 5 & 0 & 5 & 6 & 5 \\ -3 & -3 & 2 & -2 & 0 & 5 & 10 \end{bmatrix}$$

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & -1 & -3 \\ 0 & 1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 & 3 & 2 & 1 \end{bmatrix}$$

Then $B^T B = A$, $A \in \mathcal{P}(G1233)$ and $\text{rank}(A) = 3$.

Example 2.88.

$$msr(G1241) = 3$$

The tree size of $G1241$ is 4. Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 & -1 \\ 1 & 6 & 1 & 0 & 2 & 3 & -2 \\ 0 & 1 & 2 & 3 & 2 & 1 & 1 \\ 0 & 0 & 3 & 5 & 3 & 1 & 2 \\ 1 & 2 & 2 & 3 & 3 & 2 & 0 \\ 1 & 3 & 1 & 1 & 2 & 2 & -1 \\ -1 & -2 & 1 & 2 & 0 & -1 & 2 \end{bmatrix}$$

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 2 & 1 & 1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 2 & 1 & 0 & 1 \end{bmatrix}$$

Then $B^T B = A$, $A \in \mathcal{P}(G1241)$ and $\text{rank}(A) = 3$.

Example 2.89.

$$msr(G1242) = 2$$

The tree size of $G1242$ is 3. Let

$$A = \begin{bmatrix} 1 & 1 & 0 & -1 & 1 & 0 & 1 \\ 1 & 2 & 1 & -2 & 3 & -1 & 0 \\ 0 & 1 & 1 & -1 & 2 & -1 & -1 \\ -1 & -2 & -1 & 2 & -3 & 1 & 0 \\ 1 & 3 & 2 & -3 & 5 & -2 & -1 \\ 0 & -1 & -1 & 1 & -2 & 1 & 1 \\ 1 & 0 & -1 & 0 & -1 & 1 & 2 \end{bmatrix}$$

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & -1 & 1 & 0 & 1 \\ 0 & 1 & 1 & -1 & 2 & -1 & -1 \end{bmatrix}$$

Then $B^T B = A$, $A \in \mathcal{P}(G1242)$ and $\text{rank}(A) = 2$.

Example 2.90.

$$msr(G1248) = 2$$

The tree size of $G1248$ is 3. Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 1 & 2 \\ 1 & 10 & 3 & 4 & 7 & -2 & -1 \\ 0 & 3 & 1 & 1 & 2 & -1 & -1 \\ 1 & 4 & 1 & 2 & 3 & 0 & 1 \\ 1 & 7 & 2 & 3 & 5 & -1 & 0 \\ 1 & -2 & -1 & 0 & -1 & 2 & 3 \\ 2 & -1 & -1 & 1 & 0 & 3 & 5 \end{bmatrix}$$

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 1 & 2 \\ 0 & 3 & 1 & 1 & 2 & -1 & -1 \end{bmatrix}$$

Then $B^T B = A$, $A \in \mathcal{P}(G1248)$ and $\text{rank}(A) = 2$.

Example 2.91.

$$msr(G1250) = 2$$

The tree size of $G1250$ is 3. Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 2 & 1 \\ 1 & 2 & 1 & 3 & 0 & 3 & 4 \\ 0 & 1 & 1 & 2 & -1 & 1 & 3 \\ 1 & 3 & 2 & 5 & -1 & 4 & 7 \\ 1 & 0 & -1 & -1 & 2 & 1 & -2 \\ 2 & 3 & 1 & 4 & 1 & 5 & 5 \\ 1 & 4 & 3 & 7 & -2 & 5 & 10 \end{bmatrix}$$

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 2 & -1 & 1 & 3 \end{bmatrix}$$

Then $B^T B = A$, $A \in \mathcal{P}(G1250)$ and $\text{rank}(A) = 2$.

REFERENCES

- [1] R.C. Read and R. J. Wilson, *An atlas of graphs*, Oxford University Press, New York, 1998.
- [2] M.Booth, P. Hackney,, B. Harris, C.R. Johnson, M. Lay, L.H. Mitchell, S.K. Narayan, A. Pascoe, K. Steinmetz, B. D. Sutton, and W. Wang, *On the minimal rank among positive semidefinite matrices with a given graph*, SIAM J. Matrix Anal. Appl. **30**(2008), pp. 731–740.

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